

The Spin-Symmetry of the Quark Model

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Abstract

Corrections to the exact heavy-quark symmetry results are expected to follow the $1/m_Q$ mass effect of the heavy-quark. We show, by an explicit calculation, that there is something other than the mass effect that suppresses the breaking of the spin symmetry.

1 Introduction

The heavy-quark symmetry, which appears in the heavy-quark limit, gives exact results for the decays of heavy hadrons [1]. Due to the heavy-quark symmetry all form factors in the heavy-to-heavy type of decays such as $B \rightarrow D^{(*)} e \bar{\nu}_e$ ($D^{(*)} = D$ or D^*) can be related, in the heavy-quark limit, to a single universal function called the Isgur-Wise function. The Isgur-Wise function is of nonperturbative origin and has been of great interest to both theoretical and experimental studies. In the heavy-quark symmetry limit, the decoupling of the heavy-quark spin with other light fields leads to symmetry relations among hadronic matrix elements. The corrections to the symmetry are expected to follow the $1/m_Q$ mass effect of the heavy-quark. It has been unclear how well these would extrapolate to heavy-to-light quark decays, although presumably the charm quark might still be heavy enough.

Many of these symmetries were anticipated in some versions of the constituent quark model but there has not been an estimate of how much of these were an artifact of the quark model, and of the choice of wave functions. In our recent paper using the relativistic quark model [2], we found that the breaking of the spin symmetry among hadronic form factors is small even for heavy-to-light quark decays. In this paper, we show explicitly, using the same model, that there is something other than the mass effect that suppresses the breaking of the spin symmetry. In fact, the quark model keeps the spin-symmetry rules remarkably well for a wide range of masses.

We first recapitulate some aspects of spin symmetry for mesons in the heavy-quark limit. For a pseudoscalar meson $P(Q\bar{q})$ with heavy constituent quark Q , the spin of Q decouples from all other light fields in P [3]. We can therefore construct the spin operator S_Q^Z for Q such that in the heavy m_Q limit

$$S_Q^Z |P(Q\bar{q})\rangle = \frac{1}{2} |V_L(Q\bar{q})\rangle , \quad (1)$$

where $V_L(Q\bar{q})$ is the longitudinal component of a vector meson with the same quark content as P . In practice, the spin symmetry in Eq. (1) can be transformed into identities between the hadronic matrix elements, and thus some form factor relations, for $H \rightarrow P$ and $H \rightarrow V_L$, where $H(h\bar{q})$ is a pseudoscalar meson with a heavy-quark h . Using the relation [3] $[S_Q^Z, A^0 + A^3] = (1/2)(V^0 + V^3)$ for the currents $V_\mu = \bar{Q}\gamma_\mu h$ and $A_\mu = \bar{Q}\gamma_\mu\gamma_5 h$ of the transition $h \rightarrow Q$, it can be shown that Eq. (1) leads to the following identity between the hadronic matrix elements for $H \rightarrow V_L$ and $H \rightarrow P$,

$$\langle V_L | A^0 + A^3 | H \rangle = \langle P | V^0 + V^3 | H \rangle . \quad (2)$$

In $|P\rangle$ and $|V_L\rangle$ the spatial momentum of the quark Q is defined in the z -direction for the Q spinor to be an eigenstate of S_Q^Z . The spatial momenta of P and V_L should therefore also be defined in the z -direction such that the correction to the spin symmetry is of the order of Λ/m_Q , where Λ is the internal energy scale of P and V_L .

2 Kinematics

In this paper, we consider the breaking of the spin symmetry coming from a finite quark mass m_Q by directly calculating, in particular, the hadronic matrix elements in Eq. (2). We use the relativistic quark model formulated in the infinite momentum frame (or equivalently, the light-front quark model) [2, 4, 5, 6]. We first define the ratio of the matrix elements in Eq. (2) as

$$\rho(q^2) = \frac{\langle P(k') | V^0 + V^3 | H(p) \rangle}{\langle V_L(k) | A^0 + A^3 | H(p) \rangle} , \quad (3)$$

so that $1 - \rho$ represents the percentage breaking of the spin symmetry. The ratio ρ is a function of momentum transfer such that $\langle V_L | A^0 + A^3 | H \rangle$ and $\langle P | V^0 + V^3 | H \rangle$ are evaluated at the same q^2 . This allows us to evade the usual problem of kinematic discrepancies when we come to consider a number of different final states. Since the spatial momenta of P and V_L are in the z -direction, we define the function ρ in a frame where the parametrization of the momenta in the z -direction is given by

$$\begin{aligned} p^\mu &= (E_H; 0, 0, p^z) , \\ k^\mu &= (E_V; 0, 0, k^z) , \\ k'^\mu &= (E_P; 0, 0, k'^z) . \end{aligned} \quad (4)$$

The vector and scalar masses m_V and m_P are different for finite m_Q , so V_L and P will not carry the same momentum even though the initial state H has the same p^z . We write the momenta k^z and k'^z in terms of the frame parameter p^z through the condition $q^2 = (p - k)^2 = (p - k')^2$. The general parametrization in (4) includes the particular case of the infinite momentum frame in which $k^z = k'^z = p^z = P$ where $P \rightarrow \infty$ and $q^2 = 0$. It is important to note that the function ρ , when calculated in the infinite momentum frame, is defined at $q^2 = 0$ only. This is also the point of maximum recoil, which is usually difficult to treat in a non-relativistic quark model, since a large amount of energy is given to the outgoing particle. In the infinite momentum or light-front frame, we have the following connection for ρ to the form-factors defined in Ref. [2];

$$\rho^{IMF} = \frac{F_1^{H \rightarrow P}(0)}{A_0^{H \rightarrow V}(0)} . \quad (5)$$

However, here we shall calculate ρ^{IMF} directly from the matrix element.

The mass-shell conditions for p , k , and k' give the following constraints on the momenta in (4)

$$\frac{m_V}{m_P} \left(\frac{E_P + k'^z}{E_V + k^z} \right) = \frac{w - \sqrt{w^2 - 1}}{w' - \sqrt{w'^2 - 1}} , \quad (6)$$

where $w = p \cdot k / (m_H m_V)$ and $w' = p \cdot k' / (m_H m_P)$. The ratio $(E_P + k'^z) / (E_V + k^z)$ in (6) is therefore invariant for the frame defined in (4) and is a function of q^2 through the relation $q^2 = (p - k)^2 = (p - k')^2$.

It can be shown from the covariant expansion of the hadronic matrix elements [7] that $\langle P(k') | V^0 + V^3 | H(p) \rangle / (E_P + k'^z)$ and $\langle V_L(k) | A^0 + A^3 | H(p) \rangle / (E_V + k^z)$ are invariant with respect to the frame defined in (4). Using the kinematic constraint in (6), it is easy to see that the function $\rho(q^2)$ is an invariant quantity. The matching of p^z in $H \rightarrow V_L$ and $H \rightarrow P$ of ρ is the only choice that would lead to this invariance. In the definition of $\rho(q^2)$, there is an ambiguity coming from the fact that the ranges of q^2 are usually quite different in $H \rightarrow V_L$ and $H \rightarrow P$. We will therefore consider the value of ρ at $q^2 = 0$ only. The hadronic matrix elements in (3) can be calculated reliably using the relativistic quark model formulated in the infinite momentum frame. So at $q^2 = 0$, we can write

$$\rho(0) = \rho^{IMF} , \quad (7)$$

where ρ^{IMF} is calculated in the infinite momentum frame.

3 Symmetry Breaking

A brief introduction to the relativistic quark model in the infinite momentum frame can be found in Refs.[2, 5, 6]. In the relativistic quark model, the wave function for the ground state meson $M(Q\bar{q})$ is given by

$$|M(\mathbf{k})\rangle = \sqrt{2} \int d\mathbf{p}_Q \sum_{\sigma\bar{\sigma}} \Psi_{M,\sigma\bar{\sigma}}^{Jm_J} |Q(\mathbf{p}_Q, \sigma) \bar{q}(\mathbf{k} - \mathbf{p}_Q, \bar{\sigma})\rangle , \quad (8)$$

where $\mathbf{k} = P\hat{\mathbf{z}}$ is the spatial momentum of the meson M , $\mathbf{p}_Q = (\mathbf{p}_T, xP)$ and $\mathbf{p}_{\bar{q}} = \mathbf{k} - \mathbf{p}_Q = (-\mathbf{p}_T, (1-x)P)$ are those of the quarks Q and \bar{q} , respectively, in the infinite momentum frame. Here, Ψ is the momentum wave function for the $Q\bar{q}$ bound state. It has the separable form into the spin and orbital parts as $\Psi_{M,\sigma\bar{\sigma}}^{Jm_J} = R_{M,\sigma\bar{\sigma}}^{Jm_J} \phi_M$, where the expressions for $R_{M,\sigma\bar{\sigma}}^{Jm_J}$ and ϕ_M can be found in Ref. [2, 6].

In the relativistic quark model, it has been shown that ρ^{IMF} is a function of the mass ratios r , s and l , so that $\rho(0) = \rho^{IMF}(r, s, l)$, where

$$r = \frac{m_Q}{m_h} , s = \frac{m_{\bar{q}}}{m_h} , l = \frac{\Lambda}{m_h} .$$

The parameter Λ determines the internal energy scale of the meson and should be of the order of Λ_{QCD} . The dependence on l appears only in the momentum wave function and actually there could be a separate Λ for each of the mesons resulting in three further parameters. We take them all to be equal here. The kinematic region of interest for r , s , and l is such that $0 < l, r, s \leq 1$. We find,

$$\rho^{IMF} = \frac{\mathcal{I}_1}{\mathcal{I}_2} , \quad (9)$$

where

$$\mathcal{I}_1 = \int_0^1 dx \int_0^\infty dy y \phi_H \phi_P \frac{\alpha_0(1, s) \alpha_0(r, s) + y^2}{d_0(1, s) d_0(r, s)} \quad (10)$$

and

$$\mathcal{I}_2 = \int_0^1 dx \int_0^\infty dy y \phi_H \phi_V \frac{\alpha_0(1, s) \alpha_1(r, s) \alpha_2(r, s) + y^2 [\alpha_1(r, s) - \alpha_2(r, s) + \alpha_0(1, s)]}{d_0(1, s) d_1(r, s) d_2(r, s)}, \quad (11)$$

with the definitions

$$\begin{aligned} \alpha_0(r, s) &= xs + (1-x)r, \quad \alpha_1(r, s) = r + xM_0(r, s), \quad \alpha_2(r, s) = s + (1-x)M_0(r, s), \\ M_0(r, s) &= \sqrt{\frac{r^2 + y^2}{x} + \frac{s^2 + y^2}{1-x}}, \\ d_0(r, s) &= \sqrt{\alpha_0^2(r, s) + y^2}, \quad d_1(r, s) = \sqrt{\alpha_1^2(r, s) + y^2}, \quad d_2(r, s) = \sqrt{\alpha_2^2(r, s) + y^2}. \end{aligned}$$

We may write the orbital wave functions ϕ_H and $\phi_{P,V}$ in terms of a Gaussian function $\phi(r, s, l)$ such that $\phi_H = \phi(1, s, l)$ and $\phi_{P,V} = \phi(r, s, l)$. The expression for $\phi(r, s, l)$ is given by [2, 5, 6, 8]

$$\phi(r, s, l) = N \sqrt{\frac{dz}{dx}} \exp\left(-\frac{1}{2}(y^2 + z^2)/l^2\right), \quad (12)$$

where N is a normalization factor that is canceled out in ρ^{IMF} , and

$$z = (x - \frac{1}{2})M_0(r, s) - \frac{(r^2 - s^2)}{2M_0(r, s)}.$$

In Ref. [2], it has been pointed out that the scaling behavior of the meson decay constant f_M in the heavy-quark limit imposes a constraint on the orbital wave function. The Gaussian wave function in (12) is shown to satisfy the scaling law $1/\sqrt{m_Q}$ of f_M in the heavy m_Q limit.

In the numerical analysis of $\rho(0)$, it is convenient to set $s = l$ and vary s and r within the kinematic region of $0 < s, r \leq 1$. The spectator quark \bar{q} is thus considered to be a light quark with $m_{\bar{q}} = \Lambda \sim \Lambda_{\text{QCD}}$. In case of a heavy-quark decaying to a heavy or to a light quark, the corresponding regions for s and r are such that s is small and r varies between 1 and 0. By comparison, for a light quark decay to another light quark, we look at the region where s and r could both be close to 1. When $s = r = 1$, the symmetry breaking is calculated to be $1 - \rho(0) = -0.33$, using the Gaussian wave function in (12). Thus the heavy-quark symmetry does not apply in that case, as expected.

In Fig. (1), we show the plot of $1 - \rho(0)$ for a *very* heavy-quark decaying to a heavy or to a light quark so that the mass ratios s and l are very small (here they are taken to be $s = l = 0.001$). The variation of r is within the range $0 < r \leq 1$. From the Gaussian form of the wave function in Eq. (12) one expects $\langle y^2 \rangle = l^2$ and

from the expression for ρ^{IMF} in Eq. (9), we expect $1 - \rho(0) \rightarrow 0$ since $\langle y^2 \rangle \rightarrow 0$ for a very heavy decaying quark.

In Fig. (1), we show that the breaking $1 - \rho(0)$ is less than 1% in the heavy-quark limit (or small $s = 0.001$). From the figure, the mass effect of m_Q can be seen clearly as the breaking of the spin symmetry gradually increases from large r , or heavy-to-heavy decays, towards the smaller r , or heavy-to-light region. However, even at very small r , corresponding to a decay $b \rightarrow s$ ($r = 0.1$) or $b \rightarrow u$ ($r = 0.06$), it is remarkable that the symmetry breaking is less than 0.6%.

In Fig. (2), we show the plot of $1 - \rho(0)$ with the physical spectator quark mass ratios for heavy-quark decays as $s = m_{\bar{u}}/m_b = 0.06$, $s = m_{\bar{s}}/m_b = 0.1$, $s = m_{\bar{u}}/m_c = 0.2$, and $s = m_{\bar{s}}/m_c = 0.3$. In the figure, the breaking of the spin symmetry is shown to be less than about 10% for heavy b and c quark decays. In the strict heavy mass limit of $s \rightarrow 0$ as indicated in Fig. (1), the mass effect of m_Q is clearly seen as the suppression of the symmetry breaking with increasing r . Notice, however, that for finite spectator mass ratios, the function $1 - \rho(0)$ passes through zero at a recoil mass $r = m_Q/m_h$ below its heaviest limit $r = 1$.

In the case of b decays and the decay of charm to non-strange quarks, the mass effect of m_Q is still seen in the region where $1 - \rho(0)$ is positive for a large range of r . In the region where $1 - \rho(0)$ is negative, the m_Q suppression no longer follows as the size of $1 - \rho(0)$ increases with $r = m_Q/m_h$. This suggests that there are kinematic factors other than the mass effect of m_Q that govern the size of the symmetry breaking $1 - \rho(0)$, when the decaying quark has finite mass. As shown in the figure, the zero of $1 - \rho(0)$ is at smaller r when the ratio s is larger. The zero for $s \rightarrow 0$ appears at $r \rightarrow 1$ and decreases with increasing s . For $s \geq 0.251$, $1 - \rho(0)$ is negative for all r . The mass effect of m_Q is therefore less pronounced as s gets larger and the kinematic effects dominate. A different type of behavior enters for the decay involving a charm quark and a strange spectator where the deviation from the symmetry limit is rather constant and about 4%.

In fig. (3), we show the corresponding values of s and r for which $1 - \rho(0) = 0$. The zero is shown to lie within the region where s is small ($s \leq 0.251$) and the decaying quark is heavy. As shown in the figure, the zero of $1 - \rho(0)$ appears at smaller r as the mass of the decaying quark becomes less heavy (larger s). Also shown in the figure are the plots for which the spin-symmetry breaking is about 10% that is $|1 - \rho(0)| = 0.01$. It can be seen that a large portion of the possible phase space of r and s is within the region where $|1 - \rho(0)| \leq 0.01$.

We have also obtained a similar result using the harmonic oscillator wave function [4] instead of the Gaussian function. The result is therefore not an artifact of a particular momentum wave function. The quantity $\rho(0)$ has the following physical meaning:

$$|\rho(0)|^2 = \frac{(m_H^2 - m_V^2)^3}{(m_H^2 - m_P^2)^3} \frac{d\Gamma(H \rightarrow Pl\bar{\nu})/dq^2|_{q^2=0}}{d\Gamma(H \rightarrow V_L l\bar{\nu})/dq^2|_{q^2=0}}. \quad (13)$$

This allows a test of these results to be made by considering the q^2 spectrum for the semileptonic decays $H(h\bar{q}) \rightarrow P, V_L(Q\bar{q})$. The size of $\rho(0)$ for particular values of

r and s can now be measured. Repeating this for the different semileptonic decay channels of H , the dependence of $\rho(0)$ with r and s can also be determined.

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Figure 1: The plot of $1 - \rho(0)$ using the Gaussian orbital wave function. The plot is for heavy-to-heavy and heavy-to-light decays with a quark mass ratio $s = l = 0.001$ and where r varies within the range of $0 < r \leq 1$. (r refers to the ratio $r = m_Q/m_h$ in the quark decay $h \rightarrow Q$).

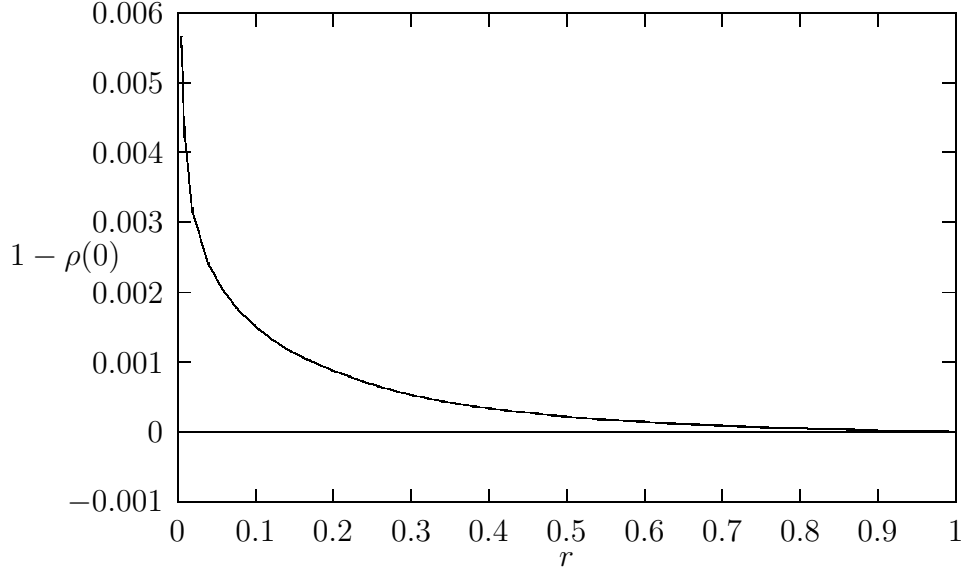


Figure 2: The plot of $1 - \rho(0)$ with physical mass ratios $s = m_{\bar{u}}/m_b = 0.06$, $s = m_{\bar{s}}/m_b = 0.1$, $s = m_{\bar{u}}/m_c = 0.2$ and $s = m_{\bar{s}}/m_c = 0.3$, for heavy b and c quark decays.

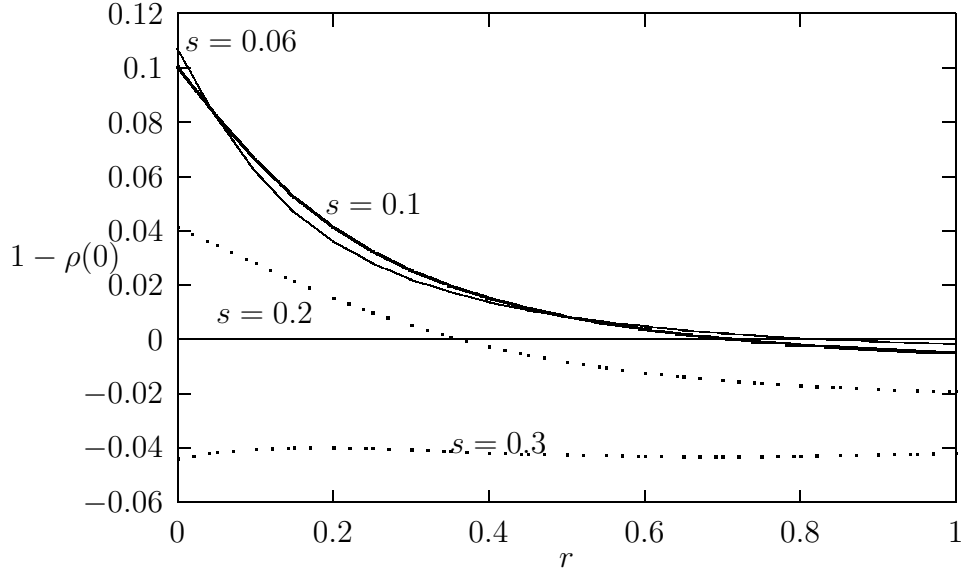


Figure 3: The corresponding values of s and r for which $1 - \rho(0) = 0$ and $1 - \rho(0) = \pm 0.01$.

